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NONAXISYMMETRIC FREE VIBRATIONS OF A SPHERICALLY ISOTROPIC SPHERICAL SHELL EMBEDDED IN AN ELASTIC MEDIUM

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Abstract—Based on three-dimensional elastic theory, the nonaxisymmetric free vibrations of a spherically isotropic spherical shell embedded in an elastic medium are studied in the paper. Three displacement functions are introduced to simplify the governing equations of a spherically isotropic medium for free vibrational problem. The Pasternak's assumption is adopted for the elastic medium, for which the P–ζ relation in the spherical coordinates is derived by the principle of minimum potential energy. It is found that the vibration of an embedded spherical shell can be divided into two classes, as the case in vacuum. The first class is identical to the corresponding one in vacuum, and the second has changed due to the effect of the surrounding medium. Numerical results are carried out to clarify the effect of relative parameters. Copyright © 1996 Elsevier Science Ltd.

1. INTRODUCTION

Problems of embedded shells and containers have been of great interest because of their wide usage in practical engineering such as underground mining, subsurface building, nuclear engineering, etc. Among them, the static and dynamic analyses of shells made of anisotropic materials are becoming more and more important with the increasing use of new types of high-strength composite materials in such areas. There are a lot of papers published that are related to problems of embedded shells. Nowiński (1957) generalized Galerkin's problem (Galerkin, 1952) to an orthotropic tube subjected to any axisymmetric temperature field. He further studied the thermoelastic problems of a spherically isotropic hollow sphere embedded in an elastic medium, which was treated as a Winkler material (Nowiński, 1959). Nath *et al.* (1987) employed Von Kármán-Donnell type nonlinear partial differential equations of motion to analyse the nonlinear dynamics of shallow shells on elastic foundation of Pasternak type. Upadhyay and Mishra (1988) dealt with the non-axisymmetric dynamic behavior of buried orthotropic cylindrical shells excited by a combination of P, SV and SH waves. Duffey and Johnson (1981) obtained the transient response of a pulsed spherical shell surrounded by an infinite elastic medium, for which the wave equation for pure radial motion was used. For an isotropic cylindrical shell buried at a depth below the free surface of the ground, Wong *et al.* (1986) have given its dynamic response from the point of view of three-dimensional elastic theory. Paliwal and Bhalla (1993) studied the large amplitude free vibrations of clamped shallow spherical shells on a Pasternak foundation using a new approach suggested by Sinharay and Banerjee (1985). For embedded spherical shell, however, to the authors' knowledge, most researches were limited to either the simplest cases, e.g. the purely radial vibration or theories based on various assumptions on the deformations of the shell. It is noted here that Cohen and Shah (1972) used two auxiliary variables and obtained two classes of vibrations for a spherically isotropic hollow sphere in vacuum on the basis of three-dimensional elastic theory. But difficulty exists in locating the expressions of displacement and stress components by their method. Earlier, Hu (1954) gave and expounded a general solution of elasticity for a spherically isotropic medium in detail. Both papers stimulate authors' interest in the study of the nonaxisymmetric free vibrations of a spherically isotropic hollow sphere embedded in an elastic medium.

Based on three-dimensional elastic theory, three displacement functions are introduced in this paper. It is found that the problem of free vibration changes to an uncoupled second-order partial differential equation and a third-order set of two coupled partial differential equations. If three displacement functions are further expanded in terms of the spherical harmonics, the original problem is finally reduced to a second-order ordinary differential equation and a second-order ordinary differential equation set. Since the uncoupled equation is a special case of the confluent hypergeometric differential equation, its solution can easily be obtained. The coupled set can be solved by Frobenius power series method. Mirsky (1964) adopted the method to solve similar equations in the analysis of free vibrations of orthotropic cylinders. Cohen and Shah (1972) also used the method to seek the solution of a second-order ordinary differential equation set. But the solutions they obtained were only special cases of the perfect solution. Recently, Ding and Chen (1995) suggested a matrix form Frobenius series method to solve such equations and got a complete solution. For surrounding elastic medium, the Pasternak model (Pasternak 1954), which includes the effect of shear interactions of the medium, is employed. The P–ζ relation is then derived out in the spherical coordinates based on the principle of minimum potential energy. It is shown that if only purely radial vibration is studied, as was done by Nowiński (1959), the Pasternak model will degenerate to the Winkler one. Considering the coupled conditions at the interface between shell and elastic medium, the frequency equations of the free vibrations of a spherically isotropic spherical shell embedded in an elastic medium can be explicitly expressed out. Effects of the relative parameters are then discussed in the present paper.

2. BASIC FORMULATIONS

For a spherically isotropic elastic medium, the spherical coordinates (r, θ, φ) are helpful with r, radial; θ, colatitudinal and φ, meridional. Assuming the center of anisotropy be identical to the origin of the coordinates, the generalized Hooke's law is:

$$\begin{cases} \sigma_\theta = A_{11}e_\theta + A_{12}e_\varphi + A_{13}e_r; & \tau_{r\theta} = A_{44}\gamma_{r\theta} \\ \sigma_\varphi = A_{12}e_\theta + A_{11}e_\varphi + A_{13}e_r; & \tau_{r\varphi} = A_{44}\gamma_{r\varphi} \\ \sigma_r = A_{13}e_\theta + A_{13}e_\varphi + A_{33}e_r; & \tau_{\theta\varphi} = \frac{1}{2}(A_{11} - A_{12})\gamma_{\theta\varphi}. \end{cases} \tag{1}$$

Here, A_{11}, A_{12}, A_{13}, A_{33}, and A_{44} are five independent elastic constants.

Since both the differential equations of motion and the strain–displacement relations of an elastic medium in the spherical coordinates can be found in Lekhnitskii (1981), they will not be repeated here. By the introduction of three displacement functions ψ, G, and w, three displacement components in the spherical coordinates, u_r, u_θ, and u_φ (in r, θ, and φ direction, respectively) can be decomposed as:

$$\begin{cases} u_r = w \\ u_\theta = -\dfrac{\partial\psi}{\partial\varphi}\operatorname{cosec}\theta - \dfrac{\partial G}{\partial\theta} \\ u_\varphi = \dfrac{\partial\psi}{\partial\theta} - \dfrac{\partial G}{\partial\varphi}\operatorname{cosec}\theta. \end{cases} \tag{2}$$

Substituting eqn (2) into the basic equations of the problem, after a long and complicate derivation, the following differential equations can be obtained:

$$\begin{cases} \frac{\partial}{\partial\theta}(A+r^2\rho\ddot{G})-\operatorname{cosec}\theta\frac{\partial}{\partial\varphi}(B-r^2\rho\ddot{\psi}) = 0 \\ \operatorname{cosec}\theta\frac{\partial}{\partial\varphi}(A+r^2\rho\ddot{G})+\frac{\partial}{\partial\theta}(B-r^2\rho\ddot{\psi}) = 0 \\ L_3w-L_4\nabla_1^2G-r^2\rho\ddot{w} = 0 \end{cases} \tag{3}$$

where ρ is the mass density of the elastic medium, and a dot over any quantity denotes its derivative with respect to time t, and

$$\begin{cases} A = L_1w-L_2G \\ B = L_5\psi \\ L_1 = (A_{13}+A_{44})\nabla_2+A_{11}+A_{12}+2A_{44} \\ L_2 = A_{44}\nabla_3^2-(2A_{44}-A_{11}+A_{12})+A_{11}\nabla_1^2 \\ L_3 = A_{33}\nabla_3^2-2(A_{11}+A_{12}-A_{13})+A_{44}\nabla_1^2 \\ L_4 = (A_{13}+A_{44})\nabla_2-A_{44}-A_{11}-A_{12}+A_{13} \\ L_5 = A_{44}\nabla_3^2-(2A_{44}-A_{11}+A_{12})+\frac{1}{2}(A_{11}-A_{12})\nabla_1^2 \\ \nabla_2 = r\frac{\partial}{\partial r} \\ \nabla_2^2 = r\frac{\partial}{\partial r}r\frac{\partial}{\partial r} \\ \nabla_3^2 = \nabla_2^2+\nabla_2 \\ \nabla_1^2 = \frac{\partial^2}{\partial\theta^2}+\cot\theta\frac{\partial}{\partial\theta}+\operatorname{cosec}^2\theta\frac{\partial^2}{\partial\varphi^2}. \end{cases} \tag{4}$$

From the first and second equation in eqns (3), we can get

$$\begin{cases} A+r^2\rho\ddot{G} = \frac{\partial H}{\partial\varphi} \\ B-r^2\rho\ddot{\psi} = \frac{\partial H}{\partial\theta}\sin\theta \end{cases} \tag{5}$$

where H should satisfy the following equation:

$$\nabla_1^2H = 0. \tag{6}$$

Generally, we can take $H \equiv 0$ without the loss of generality of the initial dynamic problem (Hu, 1954). Thus eqns (3) become

$$B-r^2\rho\ddot{\psi} = 0 \tag{7a}$$

$$A+r^2\rho\ddot{G} = 0 \tag{7b}$$

$$L_3w-L_4\nabla_1^2G-r^2\rho\ddot{w} = 0. \tag{7c}$$

It is observed that eqn (7a) is an uncoupled second-order partial differential equation in only one displacement function ψ, whereas eqn (7b) is a second-order partial differential equation and eqn (7c) is a third-order partial differential equation, and they are coupled by another two displacement functions, w and G. In order to simplify eqns (7), for the free harmonic vibration, three displacement functions ψ, G, and w are further expressed as the sum of terms below (over n from 0 to ∞):

$$[\psi, G, w] = [V(r), U(r), W(r)]S_n(\theta, \varphi)\, e^{i\omega t} \tag{8}$$

where, $S_n(\theta, \varphi)$ are the spherical harmonics, $S_n(\theta, \varphi) = P_n^m(\cos\theta)\, e^{im\varphi}$, and $P_n^m(\cos\theta)$ are the associated Legendre functions of the first kind; n, m are integers; i is the square root of -1; ω represents the circular frequency of the harmonic motion; $V(r)$, $U(r)$ and $W(r)$ are unknown functions of variable r to be solved later. (In the following, for the sake of convenience, we will write functions without their arguments, for instance, $V(r)$ will be written as V, etc.)

The following nondimensional quantities are defined:

$$\begin{cases} \bar{U} = \dfrac{U}{R}; \quad \bar{V} = \dfrac{V}{R}; \quad \bar{W} = \dfrac{W}{R}; \quad \xi = \dfrac{r}{R}; \quad t^* = \dfrac{h}{R}; \\ \Omega = \dfrac{\omega}{\omega_s} = \dfrac{\omega h}{\pi v_2}; \\ f_1 = \dfrac{A_{11}}{A_{44}}; \quad f_2 = \dfrac{A_{12}}{A_{44}}; \quad f_3 = \dfrac{A_{13}}{A_{44}}; \quad f_4 = \dfrac{A_{33}}{A_{44}}; \end{cases} \tag{9}$$

where $v_2 = \sqrt{A_{44}/\rho}$ is the elastic wave velocity, h and R are the thickness and mean radius of the spherical shell respectively. Using eqn (9) and substituting eqns (8) into eqns (7) yields:

$$\xi^2 \bar{V}'' + 2\xi \bar{V}' + \{\beta^2 \xi^2 - [2 + \tfrac{1}{2}(n^2 + n - 2)(f_1 - f_2)]\} \bar{V} = 0 \tag{10}$$

and

$$\begin{cases} \xi^2 \bar{W}'' + 2\xi \bar{W}' + (\alpha^2 \xi^2 + p_1)\bar{W} - p_2 \xi \bar{U}' - p_3 \bar{U} = 0 \\ \xi^2 \bar{U}'' + 2\xi \bar{U}' + (\beta^2 \xi^2 + p_4)\bar{U} - p_5 \xi \bar{W}' - p_6 \bar{W} = 0 \end{cases} \tag{11}$$

where prime denotes differentiation with respect to ξ, and constants $p_i (i = 1, 2, \ldots, 6)$ are given by

$$\begin{cases} p_1 = [2(f_3 - f_1 - f_2) - n(n+1)]/f_4 \\ p_2 = -n(n+1)(f_3 + 1)/f_4 \\ p_3 = n(n+1)(f_1 + f_2 + 1 - f_3)/f_4 \\ p_4 = f_1 - f_2 - 2 - n(n+1)f_1 \\ p_5 = f_3 + 1 \\ p_6 = f_1 + f_2 + 2 \\ \beta = \dfrac{\pi\Omega}{t^*}; \quad \alpha = \dfrac{\beta}{\sqrt{f_4}}. \end{cases} \tag{12}$$

Now, the fundamental dynamic equations of a spherically isotropic elastic medium are simplified to three ordinary differential equations. It is seen that eqn (10) is an independent second-order ordinary differential equation in one function $\bar{V}$. Generally, eqns (11) are coupled except for the case of $n = 0$, for which it degenerates to

$$\xi^2 \bar{W}'' + 2\xi \bar{W}' + \left[\alpha^2 \xi^2 + \frac{2(f_3 - f_1 - f_2)}{f_4}\right] \bar{W} = 0. \tag{13}$$

Solutions to eqns (10), (11), and (13) are to be given in the next section.

3. SOLUTIONS OF DIFFERENTIAL EQUATIONS

3.1. *Solutions of eqns* (10) *and* (13)

It is well known that both eqns (10) and (13) are special cases of the confluent hypergeometric differential equation, their solution can be written as:

$$\bar{V}(\xi) = \begin{cases} B_{11} j_1(\beta\xi) + B_{12} n_1(\beta\xi), & \text{if} \quad n = 1 \\ \dfrac{1}{\sqrt{\xi}} [B_{n1} J_\eta(\beta\xi) + B_{n2} Y_\eta(\beta\xi)], & \text{if} \quad n > 1 \end{cases} \tag{14}$$

where

$$\eta^2 = \tfrac{1}{4}[9 + 2(n^2 + n - 2)(f_1 - f_2)] > 0 \tag{15}$$

and

$$\bar{W}(\xi) = \frac{1}{\sqrt{\xi}} [C_{01} J_v(\alpha\xi) + C_{02} Y_v(\alpha\xi)] \tag{16}$$

where

$$v^2 = 1/4 + 2(f_1 + f_2 - f_3)/f_4 > 0 \tag{17}$$

and j_1 and n_1 are the first and second kind of the spherical Bessel functions, respectively, while J_η and Y_η are the first and second kind of the Bessel functions respectively. $B_{ij}(i = 1, 2, \ldots, j = 1, 2)$ and $C_{0i}(i = 1, 2)$ are arbitrary constants.

3.2. *Solution of eqns* (11)

When $n > 0$, eqns (11) are coupled by $\bar{U}$ and $\bar{W}$. It can be observed that the two equations of this set are of a similar form and only one regular singular point exists ($\xi = 0$). Therefore the Frobenius power series method can be applied to solve this ordinary differential equation set. The general solution of the set can be expressed by the linear combination of four independent solutions as follows (Ding and Chen, 1995):

$$\bar{U}(\xi) = \sum_{i=1}^{4} C_{nj} \bar{U}_{nj}; \quad \bar{W}(\xi) = \sum_{i=1}^{4} C_{nj} \bar{W}_{nj} \tag{18}$$

where $C_{nj}(j = 1, 2, \ldots, 4)$ are arbitrary constants, $\bar{U}_{nj}$ and $\bar{W}_{nj}(j = 1, 2, \ldots, 4)$ are a convergent, infinite series of variable ξ, and they also can be treated as functions of the circular frequency ω once the variable ξ is given.

4. PASTERNAK MODEL OF ELASTIC MEDIUM

It is worth mentioning the Winkler model of elastic foundation first. Winkler (1867) assumed the elastic foundation consisting of closely-spaced, independent linear springs. In this case, the relation between the pressure P and the deflection of the foundation surface ζ is:

$$P = k\zeta \tag{19}$$

here, k is the foundation modulus.

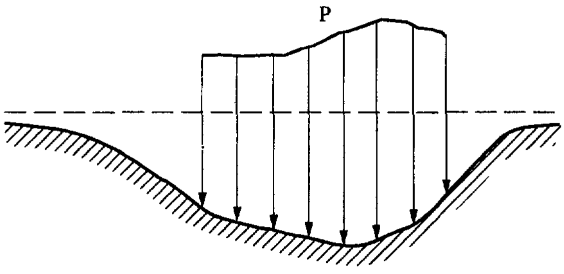

Fig. 1. Actual deformation of elastic foundation.

In fact, the deformations of the elastic foundation are always such as shown in Fig. 1. From this, Pasternak (1954) assumed the existence of shear interactions between the spring elements. This may be accomplished by connecting the ends of the springs to a beam or plate consisting of incompressible vertical elements, as given in Fig. 2, which deforms only by transverse shear.

Generally, the thickness of the shear layer in the Pasternak model used to be considered as unit (Kerr, 1964), but this is actually not the case. In the following, we will involve the effect of the thickness and, based on the basic assumption of the Pasternak model, derive the P–ζ relation by using the principle of minimum potential energy. The thickness of the shear layer is assumed to be ε, that is to say, in the present problem, the shear layer is actually a thin spherical shell with inner radius b and outer radius $b+\varepsilon$. According to the definition of the shear layer, only contributions of shear deformations to strain energy are considered, and both the effects of the displacement components in θ and φ direction on shear deformations are neglected (Pasternak, 1954). It follows that

$$\gamma_{r\theta} = \frac{\zeta_\theta}{r}; \quad \gamma_{r\varphi} = \frac{\zeta_\varphi}{r\sin\theta} \quad \text{(in shear layer)} \tag{20}$$

where

$$\zeta_\theta = \frac{\partial\zeta}{\partial\theta}; \quad \zeta_\varphi = \frac{\partial\zeta}{\partial\varphi}. \tag{21}$$

Under the pressure $P(\theta, \varphi)$, assuming ζ in the shear layer is independent of r, the total potential energy of the foundation is:

$$\prod = \int_b^{b+\varepsilon}\int_0^{\pi}\int_0^{2\pi} \frac{\bar{\mu}}{2}(\gamma_{r\theta}^2+\gamma_{r\varphi}^2)r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi$$
$$-\int_0^{\pi}\int_0^{2\pi} P\zeta b^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi + \int_0^{\pi}\int_0^{2\pi}\tfrac{1}{2}\bar{k}\zeta^2(b+\varepsilon)^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi. \tag{22}$$

Here, $\bar{\mu}$ is the shear modulus of the elastic medium and $\bar{k}$ is the spring constant.

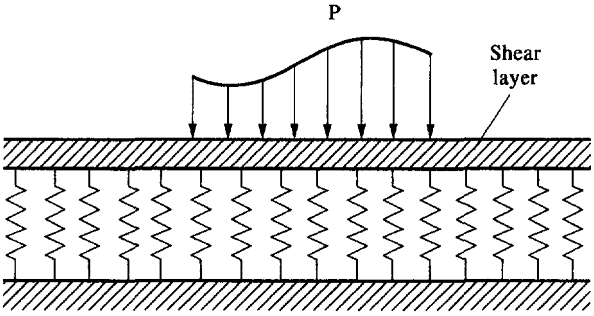

Fig. 2. Pasternak model of elastic foundation.

Substituting eqn (20) into eqn (22) gives:

$$\prod = \int_0^{\pi}\int_0^{2\pi} F(\theta, \varphi, \zeta, \zeta_\theta, \zeta_\varphi)\, d\theta\, d\varphi; \tag{23}$$

here

$$F = \tfrac{1}{2}\bar{\mu}\varepsilon(\sin\theta\zeta_\theta^2 + \operatorname{cosec}\theta\zeta_\varphi^2) - b^2 P\zeta\sin\theta + \tfrac{1}{2}(b+\varepsilon)^2\bar{k}\zeta^2\sin\theta. \tag{24}$$

The Euler equation of this functional is known to be:

$$\frac{\partial F}{\partial \zeta} - \frac{\partial}{\partial\theta}\left(\frac{\partial F}{\partial \zeta_\theta}\right) - \frac{\partial}{\partial\varphi}\left(\frac{\partial F}{\partial \zeta_\varphi}\right) = 0. \tag{25}$$

Putting eqn (24) into eqn (25) yields:

$$P = k\zeta - \frac{K}{b^2}\nabla_1^2\zeta, \tag{26}$$

where

$$k = \left(1 + \frac{\varepsilon}{b}\right)^2 \bar{k}; \quad K = \bar{\mu}\varepsilon. \tag{27}$$

In the case of spherical symmetry, ζ is independent of θ and φ, then eqn (26) degenerates to that of the Winkler one. Therefore k defined by eqn (27) are the same as that in eqn (19). Both k and K will be determined by methods such as experimental testing. Obviously, if K is zero, the medium is then reduced to the Winkler type one. It can be seen that eqn (26) is different from the conventional one (Kerr, 1964) due to the inclusion of shear layer thickness. And when the thickness is taken to be unit, this load-deformation relationship [eqn (26)] degenerates to the usual form (Kerr, 1964; Paliwal and Bhalla, 1993). In the following analysis, we assume that the bonding between the spherical shell and elastic medium is perfect and frictionless (Nath *et al.*, 1987). Namely, the vibration of the shell is affected by the elastic medium only through the normal interaction between them, i.e. the pressure P, where also the condition $\zeta = u_r$ is needed when $r = b$.

5. FREQUENCY EQUATIONS

Putting eqns (8) into the geometric relations and these in eqns (1), the expressions of stress components on the spherical surface are given by:

$$\begin{cases} \bar{\sigma}_\xi = [2\bar{W}/\xi + n(n+1)\bar{U}/\xi + (f_4/f_3)\bar{W}']S_n e^{i\omega t} \\ \bar{\tau}_{\xi\theta} = \left[(\bar{W}/\xi + \bar{U}/\xi - \bar{U}')\dfrac{\partial S_n}{\partial\theta} - (\bar{V}' - \bar{V}/\xi)\operatorname{cosec}\theta\dfrac{\partial S_n}{\partial\varphi}\right]e^{i\omega t} \\ \bar{\tau}_{\xi\varphi} = \left[(\bar{W}/\xi + \bar{U}/\xi - \bar{U}')\operatorname{cosec}\theta\dfrac{\partial S_n}{\partial\varphi} + (\bar{V}' - \bar{V}/\xi)\dfrac{\partial S_n}{\partial\theta}\right]e^{i\omega t} \end{cases} \tag{28}$$

where $\bar{\sigma}_\xi = \sigma_r/A_{13}$, $\bar{\tau}_{\xi\theta} = \tau_{r\theta}/A_{44}$, $\bar{\tau}_{\xi\varphi} = \tau_{r\varphi}/A_{44}$.

If the spherical shell is empty inside, the boundary conditions at the inner surface of the shell ($r = a$) are

$$\bar{\sigma}_\xi = \bar{\tau}_{\xi\theta} = \bar{\tau}_{\xi\varphi} = 0 \quad \left(\xi = t_1 = 1 - \frac{t^*}{2}\right). \tag{29}$$

Considering the effect of the ambient elastic medium, conditions at the outer surface of the shell ($r = b$) are:

$$\bar{\sigma}_\xi = -\bar{P}; \quad \bar{\tau}_{\xi\theta} = \bar{\tau}_{\xi\varphi} = 0 \quad \left(\xi = t_2 = 1 + \frac{t^*}{2}\right) \tag{30}$$

where $\bar{P} = P/A_{13}$. Furthermore, using eqns (28), the boundary conditions can be expressed as:

$$2\bar{W}/\xi + n(n+1)\bar{U}/\xi + (f_4/f_3)\bar{W}' = 0 \tag{31a}$$

$$(\bar{W}/\xi + \bar{U}/\xi - \bar{U}')\frac{\partial S_n}{\partial \theta} - (\bar{V}' - \bar{V}/\xi)\operatorname{cosec}\theta\frac{\partial S_n}{\partial \varphi} = 0 \tag{31b}$$

$$(\bar{W}/\xi + \bar{U}/\xi - \bar{U}')\operatorname{cosec}\theta\frac{\partial S_n}{\partial \varphi} + (\bar{V}' - \bar{V}/\xi)\frac{\partial S_n}{\partial \theta} = 0 \tag{31c}$$

for $\xi = t_1$, and

$$2\bar{W}/\xi + n(n+1)\bar{U}/\xi + (f_4/f_3)\bar{W}' + \frac{f_5 t_2 + n(n+1)f_6 t^*}{t_2^2 f_3}\bar{W} = 0 \tag{32a}$$

$$(\bar{W}/\xi + \bar{U}/\xi - \bar{U}')\frac{\partial S_n}{\partial \theta} - (\bar{V}' - \bar{V}/\xi)\operatorname{cosec}\theta\frac{\partial S_n}{\partial \varphi} = 0 \tag{32b}$$

$$(\bar{W}/\xi + \bar{U}/\xi - \bar{U}')\operatorname{cosec}\theta\frac{\partial S_n}{\partial \varphi} + (\bar{V}' - \bar{V}/\xi)\frac{\partial S_n}{\partial \theta} = 0 \tag{32c}$$

for $\xi = t_2$, where $f_5 = kb/A_{44}$ and $f_6 = K/(A_{44}h)$. Since

$$\sin\theta\frac{\mathrm{d}P_n^m}{\mathrm{d}\theta} = \frac{1}{2n+1}[n(n-m+1)P_{n+1}^m - (n+1)(n+m)P_{n-1}^m] \tag{33}$$

and noticing the orthogonal property of the Legendre functions, we can obtain following equations from eqns (31a–c) and (32a–c):

$$\bar{W}/\xi + \bar{U}/\xi - \bar{U}' = 0 \quad (\xi = t_1 \text{ or } t_2) \tag{34a}$$

$$\bar{V}' - \bar{V}/\xi = 0 \quad (\xi = t_1 \text{ or } t_2) \tag{34b}$$

From boundary conditions (31a), (32a) and (34a,b) and from differential equations (10), (11) and (13), we reach a conclusion that the free vibrations of a spherically isotropic spherical shell embedded in an elastic medium can be divided into two classes, as the case in vacuum (Cohen and Shah, 1972). The first class is defined by eqn (10) and condition (34b), while the second by eqns (11) or (13) and conditions (31a), (32a) and (34a). It is shown that the first class, which corresponds to a equivoluminal motion of the shell, is characterized by the absence of radial component of displacement while for the second class, the displacement has, in general, both transverse and radial components, but the rotation has no radial component. Since the first class of vibration is exactly identical to that in vacuum, we will not repeat it here. Details of this class can be found in Cohen and Shah (1972), for example.

Using the results obtained in the preceeding sections and allowing for the boundary conditions, we can obtain some systems of linear algebra equations with different sets of arbitrary constants. It is well known that, for each system, there exist nontrivial solutions only when the determinant of the coefficient matrix vanishes. Therefore, the following frequency equations are then obtained (for the second class only).

5.1. *When* n = 0

The corresponding frequency equation is:

$$|E^1_{ij}| = 0 \quad (i,j = 1,2) \tag{35}$$

where

$$\begin{cases} E^1_{11} = [2+(v-1/2)(f_4/f_3)]J_v(\alpha t_1)-(f_4/f_3)\alpha t_1 J_{v+1}(\alpha t_1), \\ E^1_{12} = [2+(v-1/2)(f_4/f_3)]Y_v(\alpha t_1)-(f_4/f_3)\alpha t_1 Y_{v+1}(\alpha t_1), \\ E^1_{21} = [2+(v-1/2)(f_4/f_3)]J_v(\alpha t_2)-(f_4/f_3)\alpha t_2 J_{v+1}(\alpha t_2)+(f_5/f_3)J_v(\alpha t_2), \\ E^1_{22} = [2+(v-1/2)(f_4/f_3)]Y_v(\alpha t_2)-(f_4/f_3)\alpha t_2 Y_{v+1}(\alpha t_2)+(f_5/f_3)Y_v(\alpha t_2). \end{cases} \tag{36}$$

It is noted that eqn (35) is, in fact, the corresponding frequency equation of the purely radial vibration of the embedded spherical shell.

5.2. *When* n > 0

The frequency equation is obtained as:

$$|E^2_{ij}| = 0 \quad (i,j = 1,2\ldots 4) \tag{37}$$

where

$$\begin{cases} E^2_{1i} = 2\bar{W}_{ni}(t_1)/t_1+n(n+1)\bar{U}_{ni}(t_1)/t_1+(f_4/f_3)\bar{W}'_{ni}(t_1), \\ E^2_{2i} = \bar{W}_{ni}(t_1)/t_1+\bar{U}_{ni}(t_1)/t_1-\bar{U}'_{ni}(t_1), \\ E^2_{3i} = \bar{W}_{ni}(t_2)/t_2+\bar{U}_{ni}(t_2)/t_2-\bar{U}'_{ni}(t_2), \\ E^2_{4i} = 2\bar{W}_{ni}(t_2)/t_2+n(n+1)\bar{U}_{ni}(t_2)/t_2+(f_4/f_3)\bar{W}'_{ni}(t_2) \\ \qquad +\dfrac{f_5 t_2+n(n+1)f_6 t^*}{f_3 t_2^2}\bar{W}_{ni}(t_2). \end{cases} \quad (i = 1,2\ldots 4) \tag{38}$$

Obviously, if both parameters of the ambient elastic medium k and K are taken to be zero, then eqns (35) and (37) reduce to the corresponding frequency equation of free vibration in vacuum (Cohen and Shah, 1972). It is also noted that the frequency equations will degenerate to the corresponding ones of an embedded isotropic spherical shell upon the following substitution:

$$A_{11} = A_{33} = \lambda+2\mu; \quad A_{12} = A_{13} = \lambda; \quad A_{44} = \mu \tag{39}$$

where, λ and μ are Lamé constants.

It is noted that, Silbiger (1962) has stated in detail that the nonaxisymmetric modes of vibrations of thin isotropic spherical shells can be obtained by the superposition of axisymmetric ones of identical natural frequency. Since spherical isotropy does not violate the spherical symmetry of the shell, as shown above, the integer m, which appears in the spherical harmonics and represents the nonaxisymmetric motion ($m \neq 0$) of the shell, is not included in the frequency equations.

Table 1. Elastic constants of two anisotropic materials

Material	f_1	f_2	f_3	f_4
Material A	3.64	1.60	1.32	3.76
Material B	20.00	12.00	2.00	2.00

6. NUMERICAL RESULTS AND DISCUSSION

For the second class of vibration of a spherically isotropic spherical shell embedded in an elastic medium of Pasternak type, calculations are carried out to clarify the effects of various parameters involved in the frequency equations. It is shown that the nondimensional frequency Ω is only related to the nondimensional parameters f_1-f_6 and t^*. In our calculations, two spherically isotropic materials are considered, for which the elastic constants are listed in Table 1. Material A is nearly isotropic like magnesium, while Material B is a hypothetical one, exhibiting substantial anisotropy. Numerical results are given in the form of figures. It is noted that, although for each frequency equation there are more than one roots, only the smallest positive root for given parameters, which is of physical significance, is given in figures for $n = 0$ as well as $n > 0$.

6.1. n = 0

Calculations are first made for $n = 0$, the purely radial vibration or the breathing mode. In this case, the nondimensional elastic foundation constant f_6 has no effect on the vibration of the spherical shell, which can be observed by looking into the frequency equation (35). For each material, results are given for four values of thickness-to-mean radius ratio (t^*). Spectra of nondimensional frequency Ω vs nondimensional elastic foundation constant f_5 are shown in Figs 3 and 4.

From the results, it is shown that the nondimensional frequency Ω increases when the nondimensional foundation constant f_5 grows. It is seen that each curve displayed in Figs 3 and 4 is similar to the upper branch of a parabola. It actually somehow resembles a spring–mass system of single freedom degree, for which the curve of frequency vs spring constant is known to be a parabola, when the mass of the system is given. Since the spherical shell has stiffness itself, its free vibrational frequency does not equal zero when f_5 is taken to be zero. It is also shown that the nondimensional frequency Ω increases when the thickness-to-mean radius ratio t^* of the shell increases. If the spherical shell and the surrounding elastic medium can actually be modeled as a single spring–mass system, it will

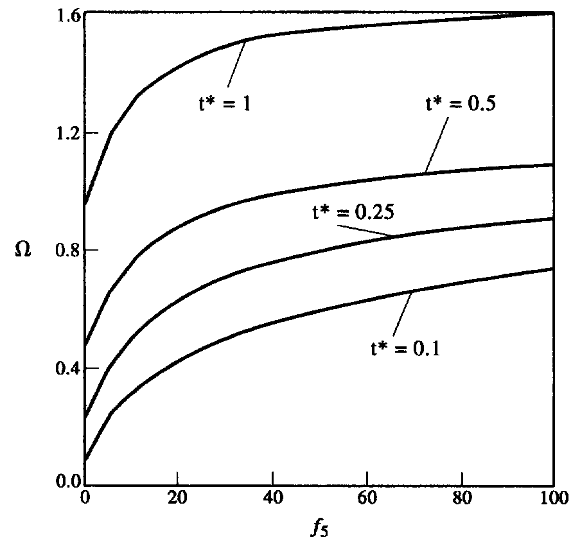

Fig. 3. Spectra of nondimensional frequency Ω vs nondimensional elastic foundation constant f_5 for breathing mode (Material A).

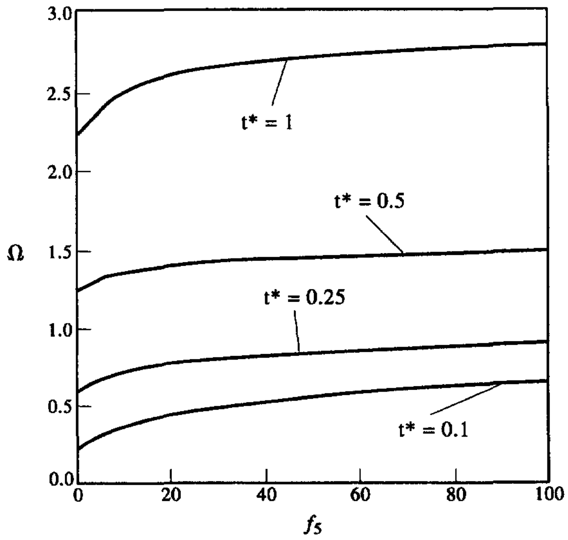

Fig. 4. Spectra of nondimensional frequency Ω vs nondimensional elastic foundation constant f_5 for breathing mode (Material B).

indicate that the effective stiffness of the system grows more rapidly than its effective mass. Moreover, the nondimensional frequency is greatly affected by the elastic constants of the shell as Figs 3 and 4 show us. Results for Material A are lower than those for Material B to some degree. This may be of great importance in structural design of practical engineering. Especially, in the areas that have severe restrictions on dynamics of the structures, the results of this example illustrate that new types composite materials may provide such demanded conditions.

6.2. $n > 0$

In this case, the displacement of the shell has both transverse and radial components as indicated in the preceding part. Therefore, unlike the breathing mode, these non-breathing modes are affected by both the nondimensional foundation constants f_5 and f_6. For comparison purposes, calculations are carried out both for Material A and Material B. For each material, two ratios of thickness-to-mean radius are involved. Results are displayed in Figs 5–8. In each figure, three curves of nondimensional frequency corresponding to mode number $n = 1$, 2, and 3 are given.

Figure 5 displays the spectra of nondimensional frequency Ω vs the nondimensional elastic foundation constant f_5 for Material A with four different values of the other nondimensional foundation constant f_6 given in the legend. When f_6 is taken to be zero, the elastic medium is, in fact, of the Winkler type, and the corresponding results are given in Fig. 5(a). When it is nonzero, the elastic medium is modeled as the Pasternak type and results are given in Figs 5(b,c). Three modes corresponding to $n = 1$, 2, and 3 are presented. By comparing Fig. 5(a) with Fig. 5(b,c), it can be seen that there is an obviously different point between the Winkler type medium and the Pasternak one. Especially, when f_5 equals zero, the frequency of spherical shell embedded in the Winkler type elastic medium for $n = 1$ is zero, as Fig. 5(a) shows, while it is not zero when the elastic medium is considered as the Pasternak type one, as Figs 5(b,c) show. In fact, the case corresponds to a spherical shell vibrating in vacuum when the elastic medium is treated as the Winkler type and the foundation constant f_5 is taken to be zero, for which there exists a rigid movement of the spherical shell when $n = 1$. We can also observe from Fig. 5(a) that the nondimensional frequency stays basically invariable when the foundation constant f_5 is a large value (e.g. $\geqslant 40$). Also, the spectra curve is nearly horizontal when the foundation constant f_6 is large, as Fig. 5(c) shows us. An obvious trend in that frequency for higher mode number n is always larger than the corresponding one for lower n, is also obtained. Spectra curves for higher mode numbers ($n \geqslant 4$) are not presented here because they are all similar to $n = 2$

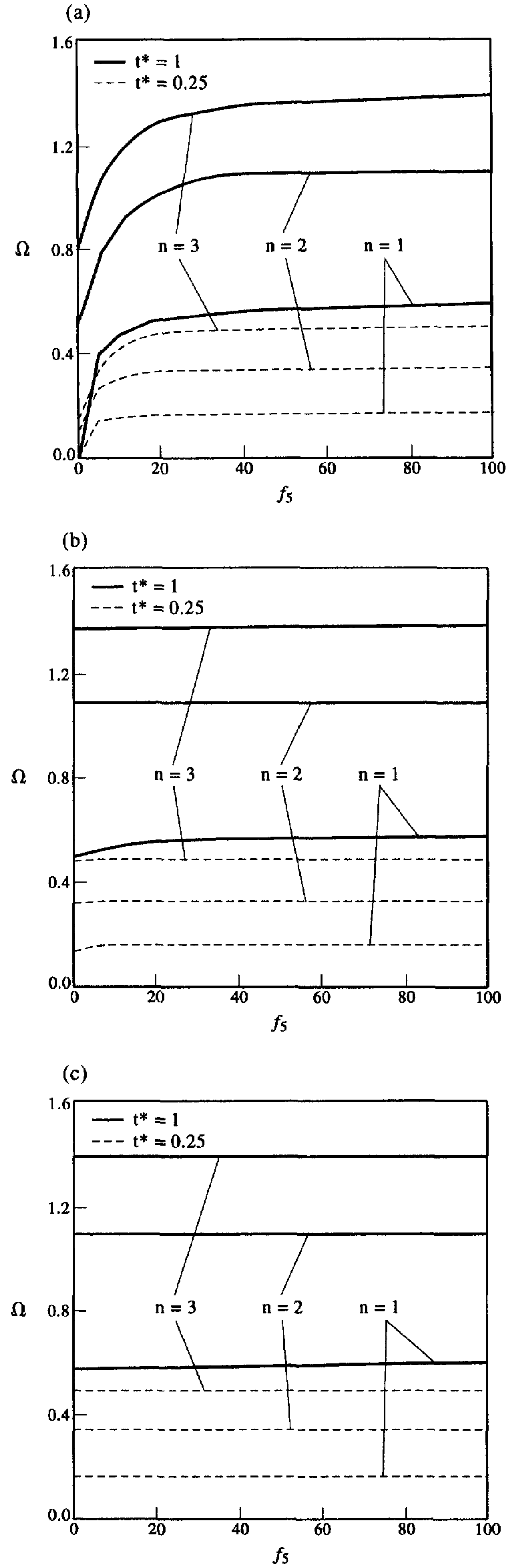

Fig. 5. Spectra of nondimensional frequency Ω vs nondimensional elastic foundation constant f_5 for $n = 1, 2, 3$ (Material A). (a) $f_6 = 0$. (b) $f_6 = 10$. (c) $f_6 = 50$.

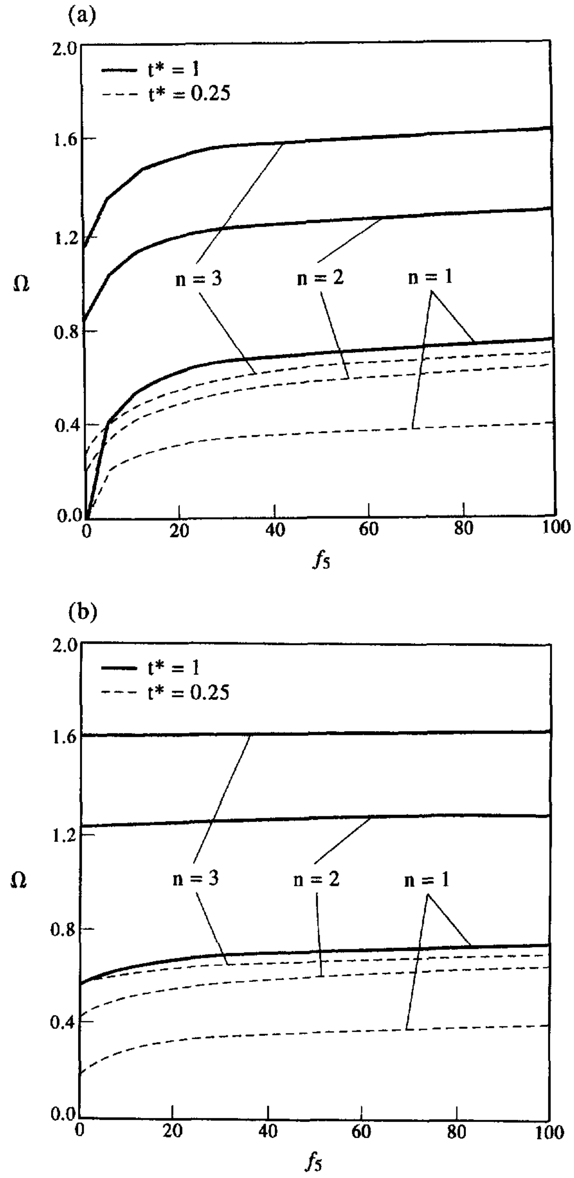

Fig. 6. Spectra of nondimensional frequency Ω vs nondimensional elastic foundation constant f_5 for $n = 1, 2, 3$ (Material B). (a) $f_6 = 0$. (b) $f_6 = 10$.

or 3 except for larger frequency values. By comparison of the actual and dotted lines in Fig. 5, we can see that the frequencies for larger ratio of thickness-to-mean radius ($t^* = 1$) are larger than those for the lower one ($t^* = 1/4$) as that is observed for $n = 0$ (Figs 3 or 4).

Figure 6 displays the spectra of nondimensional frequency Ω vs the elastic foundation constant f_5 for Material B. Since for larger f_6 the spectra stay nearly invariable, as indicated above, only two typical values of f_6 (0 and 10) are considered and the corresponding spectra curves are given in Figs 6(a,b). Though the frequencies of Material B are higher than the corresponding ones of Material A, there is actually no great difference between them. Therefore, all points for results of Material A mentioned above can also be applied for those of Material B and we do not repeat them again. Probably, the frequencies for $t^* = 1$ are higher than those for $t^* = 1/4$.

To clarify the effect of another foundation constant f_6, calculations are then made to give spectra of nondimensional frequency Ω versus f_6. Figure 7 displays the spectra of Ω vs f_6 for Material A for two values of f_5 which are given in the legend, while Fig. 8 shows

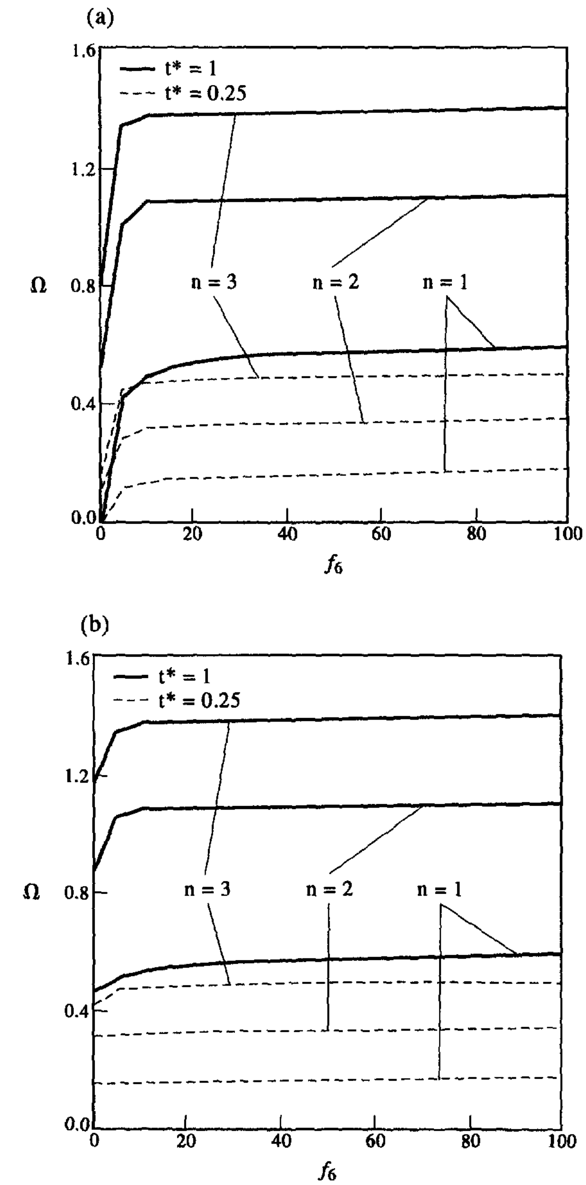

Fig. 7. Spectra of nondimensional frequency Ω vs nondimensional elastic foundation constant f_6 for $n = 1, 2, 3$ (Material A). (a) $f_5 = 0$. (b) $f_5 = 10$.

those for Material B. Since f_6 has a similar effect to f_5 in the frequency equation, as pointed out earlier, similar observations are also obtained from Figs 7 and 8.

7. CONCLUSION

This paper presents an analytic three-dimensional elastic method to analyse the non-axisymmetric free vibrations of a spherically isotropic spherical shell embedded in an elastic medium. Therefore, it provides a basis for checking the capability of different shell theories on this problem. It is further shown that the vibrations of embedded spherical shells can be divided into two classes, as the case in vacuum. The first class is identical to the corresponding one in vacuum, and the second has changed due to the effect of the surrounding medium.

The elastic medium is taken to be the Pasternak type and based on its basic assumptions we rederive the pressure deflection relation by using the principle of minimum potential

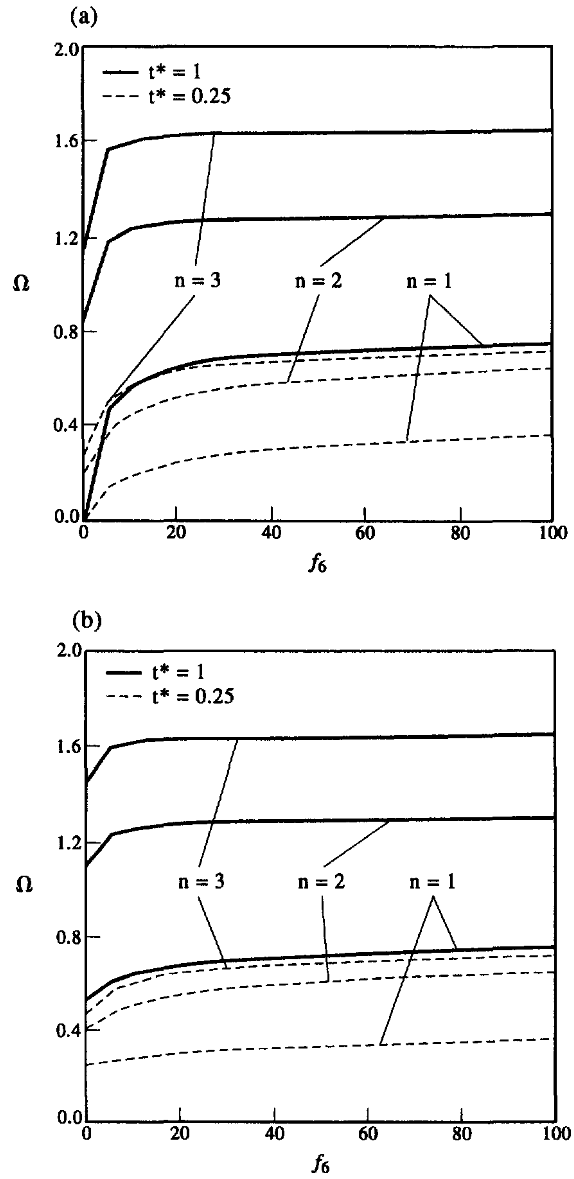

Fig. 8. Spectra of nondimensional frequency Ω vs nondimensional elastic foundation constant f_6 for $n = 1, 2, 3$ (Material B). (a) $f_5 = 0$. (b) $f_5 = 10$.

energy. Though the resulting formula is similar to that presented by Pasternak (1954), it is found that more subtle effects are brought by the shear layer thickness.

One point that should be mentioned is that, for the second class, the embedded spherical shell has zero vibrational frequency when the mode number $n = 1$ and when both foundation constants are taken to be zero, for which the shell is, in fact, in vacuum. The case corresponds to the rigid movement of the shell. Since the two foundation constants have similar effects in the frequency equation [eqn (37)], their results are actually the same as Figs 5–8 show us.

From the results, we can also see that for large value of f_6 or f_5, the spectra curve of Ω vs f_5 or f_6 is nearly a horizontal line. That is to say, the frequency stays basically invariable when f_6 or f_5 is large.

It is seen from the results, that both the elastic constants of the spherical shell and the ratio of thickness-to-mean radius of the shell have significant effects on the free vibration frequency of embedded spherical shells. This may be of great importance in practical engineering.

Though there are works related to free vibration problems of embedded cylindrical shell [see Men and Yuan (1990), for example], it is very regretable that no available works on spherical shells can be found by the authors, namely, comparisons cannot be made with results obtained from relevant literature. Therefore, shell theory analysis of embedded spherical shells is needed and will be presented in another paper.

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